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# Special function solutions for asymmetric discrete Painlevé equations

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**Abstract.** We study the solutions of a family of discrete Painlevé equations. The equations that we examine are given as a system of two first-order non-autonomous mappings. The solutions we are interested in are the ones obtained whenever the Painlevé equation can be reduced to a discrete Riccati equation, which can be linearized through a Cole–Hopf transformation. The special solutions thus obtained involve generalizations or reductions of the hypergeometric (and $q$-hypergeometric) function.

## 1. Introduction

One of the special properties of the Painlevé equations is the fact that they possess solutions in terms of special functions [1]. These solutions do not represent the full general solution of these ordinary differential equations, which, as is well known, introduces new transcendents [2]. In fact they correspond to a one-degree-of-freedom solution for a particular set of parameters. Let us give an example here. Consider the $P_{II}$ equation

$$w'' = 2w^3 + zw + \mu. \tag{1.1}$$

Whenever $\mu$ is of the form $\mu = \frac{1}{2} + m$ with $m \in \mathbb{Z}$, a subset of solutions of (1.1) can be expressed in terms of Airy functions. In particular for $\mu = \frac{1}{2}$ we have $w = -qA'(qz)/A(qz)$ where $A$ satisfies the Airy equation $A''(u) = uA(u)$ and $q = -1/\sqrt[3]{2}$. For higher values of $\mu$ the solution can be constructed using the Schlesinger transformations of $P_{II}$ and it can be shown that it can be expressed in terms of a Wronskian determinant, the elements of which are Airy functions (and their derivatives).

Discrete Painlevé equations [3] have the same properties as their continuous counterparts. Let us illustrate this by the discrete analogue of the example we presented above. We start with the discrete $P_{II}$ equation:

$$\overline{x} + \underline{x} = \frac{zx + \mu}{1 - x^2} \tag{1.2}$$

where $z \equiv z(n) = \alpha n + \beta$, $x \equiv x(n)$, $\overline{x} = x(n+1)$, $\underline{x} = x(n-1)$. Here when $\mu = \alpha/2$ the solution of (1.2) can be expressed as $x = \overline{A}/A - 1$ where $A(n)$ obeys the mapping

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$\overline{\overline{A}} - 2\overline{A} + (z + \alpha/2)A = 0$. This is the discrete analogue of the Airy equation [4]. For higher values of $\mu$ of the form $\mu = (\frac{1}{2} + m)\alpha$ it can be shown that the solutions can be expressed in terms of Casorati determinants involving discrete Airy functions. The special function solutions of a large class of discrete Painlevé equations have already been studied in [5].

In this paper we will address the problem of the special function solutions of *asymmetric* discrete Painlevé equations. The term 'asymmetric' needs some explanation. When the QRT [6] mapping was proposed, it was presented under two forms: a one-component one called 'symmetric' and a two-component one called asymmetric. We have carried over this terminology to the discrete Painlevé equations (but we are conscious that it is not a very appropriate one). An asymmetric discrete Painlevé equation in this paper will designate a mapping of the form:

$$\underline{x} = \frac{f_1(y) - xf_2(y)}{f_4(y) - xf_3(y)} \tag{1.3a}$$

$$\overline{y} = \frac{g_1(x) - yg_2(x)}{g_4(x) - yg_3(x)} \tag{1.3b}$$

(discrete Painlevé equations involving more than two dependent variables do exist [7], and have been identified but they will not be the object of the present study).

The asymmetric discrete Painlevé equations we shall consider are interconnected in what we call a coalescence cascade, which means that starting from some equation with a high number of parameters one can obtain the ones with lower number of parameters through a degeneration procedure. To be more specific, the equations we shall examine are arranged in the following degeneration pattern:

$$\begin{array}{ccccc}
\alpha q\text{-P}_{\text{V}} & \longrightarrow & \alpha\text{d-P}_{\text{IV}} & & \\
\downarrow & & \downarrow & & \\
\alpha q\text{-P}_{\text{III}} & \longrightarrow & \text{d-P}_{\text{V}} & \longrightarrow & \text{d-P}_{\text{IV}} \\
 & & \downarrow & & \downarrow \\
 & & \alpha\text{d-P}_{\text{II}} & \longrightarrow & \alpha\text{d-P}_{\text{I}}.
\end{array}$$

The names of the equations will become explicit in the next section. Here the symbol $\alpha$ stands for asymmetric. The two types of d- and $q$-equations distinguish the way in which the independent variable enters. For d-equations it enters in an additive way, i.e. $z = \alpha n + \beta$, while for a $q$-equation it enters in a multiplicative way, i.e. $z = z_0\lambda^n$. The above pattern does not exhaust all the possible degenerations, as we have shown in [8], but the remaining lower equations possess symmetric forms (and have already been studied [9]).

How does one obtain a solution in terms of special functions for a discrete Painlevé equation? Following the analogy with the continuous Painlevé equations we seek a solution that satisfies a discrete Riccati in one of the variables:

$$x = \frac{a\underline{x} + b}{c\underline{x} + d} \tag{1.4a}$$

while the two variables are related through a homographic transformation

$$y = \frac{fx + g}{hx + j}. \tag{1.4b}$$

(All the parameters $a, b, c, d, f, g, h, j$ may, of course, depend on the independent variable $n$.) Once the Riccati equation is obtained, one can linearize it through the use of a Cole–Hopf transformation simply by putting $x = H/G$. This leads to a three-point linear mapping which is the discrete equivalent of the linear second-order equations of the hypergeometric family. The examples that will be presented in what follows will make these notions clearer.

## 2. The construction of the discrete special function solutions

As we explained in the introduction, the asymmetric discrete Painlevé equations we shall examine are organized in a coalescence cascade where a given 'higher' equation leads to one (or more) 'lower' ones through a limiting procedure involving the dependent and independent variables as well as the parameters. In order to make the presentation clearer we will use the following convention. *A priori*, the equations will be expressed in lower-case letters: $x, y, z$ etc. For the coalescence procedure involving two equations, the higher equation variables will be written in upper-case letters: $X, Y, Z$, etc, and the lower equation ones in lower-case letters. The small parameter used in the coalescence procedure will be represented by $\delta$ while, whenever continuous limits are considered, the corresponding small parameter will be $\epsilon$.

### 2.1. *The asymmetric* $q$-$\mathrm{P_V}$

We start with the asymmetric $q$-$\mathrm{P_V}$ equation which is the name of the system:

$$(yx-1)(y\underline{x}-1)=\frac{(y-u)(y-v)(y-w)(y-s)}{(1-py/z)(1-y/pz)} \tag{2.1a}$$

$$(yx-1)(\overline{y}x-1)=\frac{(x-1/u)(x-1/v)(x-1/w)(x-1/s)}{(1-rx/\tilde{z})(1-x/r\tilde{z})} \tag{2.1b}$$

with the constraint $uvws=1$ and where $z=z_0\lambda^n$ and $\tilde{z}=z_0\lambda^{n+1/2}$. For future convenience we introduce the parameter $\mu=\lambda^{1/2}$. The linearization of (2.1) can be obtained most simply by the splitting procedure we have introduced in [10]. Namely we split each of the equations of the system in two parts and we request that the resulting system:

$$yx-1=-\frac{(y-u)(y-v)}{uv(1-py/z)} \tag{2.2a}$$

$$y\underline{x}-1=-\frac{(y-w)(y-s)}{(1-y/pz)ws} \tag{2.2b}$$

$$yx-1=-\frac{uv(x-1/u)(x-1/v)}{(1-rx/\tilde{z})} \tag{2.2c}$$

$$\overline{y}x-1=-\frac{ws(x-1/w)(x-1/s)}{(1-x/r\tilde{z})} \tag{2.2d}$$

be compatible. The condition for compatibility is

$$r=\mu uvp. \tag{2.3}$$

This is precisely the linearizability condition. Indeed when (2.3) holds we can obtain form (2.2) a homographic mapping for $x$ in the form

$$x=\frac{\underline{x}sw(puv-(u+v)z+pz^2)-(p^2uv-sw)z+p(u+v-s-w)z^2}{\underline{x}z(p^2-1)+puv(sw-pz(s+w)+z^2)}. \tag{2.4}$$

This discrete Riccati equation can be linearized through a Cole–Hopf transformation. The resulting equation lies beyond the hypergeometric equation just as asymmetric $q$-$\mathrm{P_V}$ goes beyond $\mathrm{P_{VI}}$ (since it has one extra parameter). However, as we have shown in [11], asymmetric $q$-$\mathrm{P_V}$ *does* go over to $\mathrm{P_{VI}}$ at the continuous limit. Indeed taking $u=\theta e^{\epsilon a}$, $v=\theta^{-1}e^{\epsilon b}$, $w=\theta e^{-\epsilon a}$, $s=\theta^{-1}e^{-\epsilon b}$, $\lambda=e^{\epsilon}$, $p=e^{\epsilon c}$, $r=e^{\epsilon d}$, $\omega=(x-\theta)/(\theta^{-1}-\theta)$, $\zeta=(z-\theta)/(\theta^{-1}-\theta)$, $y=(z(x-\theta^{-1}-\theta)+1)/(x-z)+\epsilon\psi$, where the constraint $uvws=1$ has been implemented, we obtain after eliminating $\psi$ from two first-order equations:

$$\frac{d^2\omega}{d\zeta^2}=\frac{1}{2}\left(\frac{1}{\omega}+\frac{1}{\omega-1}+\frac{1}{\omega-\zeta}\right)\left(\frac{d\omega}{d\zeta}\right)^2-\left(\frac{1}{\zeta}+\frac{1}{\zeta-1}+\frac{1}{\omega-\zeta}\right)\frac{d\omega}{d\zeta}$$

$$+\frac{\omega(\omega-1)(\omega-\zeta)}{2\zeta^2(\zeta-1)^2}\left(A+\frac{B\zeta}{\omega^2}+\frac{C(\zeta-1)}{(\omega-1)^2}+\frac{D\zeta(\zeta-1)}{(\omega-\zeta)^2}\right) \tag{2.5}$$

i.e. precisely $\mathrm{P_{VI}}$ where $A=4c^2$, $B=-4b^2$, $C=4a^2$ and $D=1-4d^2$.

The same continuous limit on the discrete Riccati (2.4), leads to the continuous Riccati:

$$\zeta(1-\zeta)\frac{\mathrm{d}\omega}{\mathrm{d}\zeta}=2d\omega^2+(2(a+b)\zeta-2a-2c)\omega-2b\zeta \tag{2.6}$$

where the linearization condition is now $d=a+b+c+\frac{1}{2}$, and the Cole–Hopf transformation $\omega=\zeta-\frac{\zeta(1-\zeta)}{2cG}\frac{dG}{d\zeta}$ linearizes it to

$$\zeta(1-\zeta)\frac{\mathrm{d}^2G}{\mathrm{d}\zeta^2}+(2a+2c+1-(2c+2d+1)\zeta)\frac{\mathrm{d}G}{\mathrm{d}\zeta}-4cdG=0 \tag{2.7}$$

i.e. the Gauss-hypergeometric equation in canonical form.

The coalescence procedure applied to asymmetric $q$-$\mathrm{P_V}$ allows one to obtain either asymmetric d-$\mathrm{P_V}$ or asymmetric $q$-$\mathrm{P_{III}}$. Let us study the first limit.

### *2.2. The asymmetric* d-$\mathrm{P_{IV}}$

In order to obtain asymmetric d-$\mathrm{P_{IV}}$ starting from asymmetric $q$-$\mathrm{P_V}$ we introduce the following transformation: $X=1+\delta x$, $Y=1+\delta y$, $Z=1+\delta z$, $\lambda=1+\delta\alpha$, $U=1+\delta u$, $V=1+\delta v$, $W=1+\delta w$, $S=1+\delta s$, $P=1+\delta p$, $R=1+\delta r$ where now $z=\alpha n+\beta$. At the limit $\delta\to 0$ we obtain the system:

$$(y+x)(y+\underline{x})=\frac{(y-u)(y-v)(y-w)(y-s)}{(y+p-z)(y-p-z)} \tag{2.8a}$$

$$(y+x)(\overline{y}+x)=\frac{(x+u)(x+v)(x+w)(x+s)}{(x+r-\tilde{z})(x-r-\tilde{z})} \tag{2.8b}$$

with the constraint $u+v+w+s=0$ and where $\tilde{z}=z+\alpha/2$. Instead of performing the linearization splitting from the start, we use the coalescence limit on asymmetric $q$-$\mathrm{P_V}$. We find thus the system:

$$y+x=\frac{(y-u)(y-v)}{(y+p-z)} \tag{2.9a}$$

$$y+\underline{x}=\frac{(y-w)(y-s)}{(y-p-z)} \tag{2.9b}$$

$$y+x=\frac{(x+u)(x+v)}{(x+r-\tilde{z})} \tag{2.9c}$$

$$\overline{y}+x=\frac{(x+w)(x+s)}{(x-r-\tilde{z})} \tag{2.9d}$$

and the compatibility–linearizability condition reads

$$r=u+v+p+\alpha/2. \tag{2.10}$$

The discrete Riccati reads

$$x=\frac{\underline{x}((z+p)(z-p-u-v)+uv)+(z-p-u-v)(sw-uv)-2puv}{2p\underline{x}+(z-p)(z+p+u+v)+sw}. \tag{2.11}$$

Its linearization leads again to a discrete linear equation that goes, in principle, beyond the hypergeometric. As in the case of asymmetric $q$-$\mathrm{P_V}$ the continuous limit can be easily obtained. For asymmetric d-$\mathrm{P_{IV}}$ it leads to $\mathrm{P_{VI}}$ [11]. We put $u=\frac{1}{2}+\epsilon a$, $v=-\frac{1}{2}+\epsilon b$, $w=\frac{1}{2}-\epsilon a$,

$s = -\frac{1}{2} - \epsilon b$, $p = \epsilon c$, $r = \epsilon d$, $x = \omega - \frac{1}{2}$, $z = \zeta - \frac{1}{2}$, $y = \omega(\zeta - 1)/(\omega - \zeta) + \frac{1}{2} + \epsilon\psi$, and after again eliminating $\psi$ in two first-order equations we recover at $\epsilon \to 0$ the equation (2.5) with $A = 4c^2$, $B = -4a^2$, $C = 4b^2$ and $D = 1 - 4d^2$. The same approach on the Riccati, where the linearization condition is now $d = a + b + c + \frac{1}{2}$, leads to a continuous equation linearized with the same Cole–Hopf as (2.6) to

$$\zeta(1-\zeta)\frac{\mathrm{d}^2 G}{\mathrm{d}\zeta^2} + (\mathrm{d} - a - (2c + 2d + 1)\zeta)\frac{\mathrm{d}G}{\mathrm{d}\zeta} - 4cdG = 0 \tag{2.12}$$

again the hypergeometric equation.

### 2.3. *The asymmetric* $q$-$\mathrm{P_{III}}$

The asymmetric $q$-$\mathrm{P_V}$ has another coalescence limit to the asymmetric $q$-$\mathrm{P_{III}}$ equation. Putting: $X = x/\delta$, $Y = y/\delta$, $Z = z/\delta$, $U = u/\delta$, $V = v\delta$, $W = w/\delta$, $S = s\delta$, $P = p$, $R = r$, we find at $\delta \to 0$ the mapping

$$x\underline{x} = \frac{(y-u)(y-w)}{(1-yp/z)(1-y/pz)} \tag{2.13$a$}$$

$$y\overline{y} = \frac{(x-1/v)(x-1/s)}{(1-xr/\tilde{z})(1-x/r\tilde{z})} \tag{2.13$b$}$$

with the obvious condition $uvws = 1$. Equation (2.13) can be written in canonical form by introducing a gauge $y \to zy$, $x \to \tilde{z}x$. We obtain thus

$$x\underline{x} = \frac{(y-u/z)(y-w/z)}{(1-yp)(1-y/p)} \tag{2.14$a$}$$

$$y\overline{y} = \frac{(x-1/\tilde{z}v)(x-1/\tilde{z}s)}{(1-xr)(1-x/r)}. \tag{2.14$b$}$$

Equation (2.14) was studied by Jimbo and Sakai [12] who have shown that it is a $q$-discrete form of $\mathrm{P_{VI}}$. Thus, this equation is often referred to as the $q$-$\mathrm{P_{VI}}$ equation. Its linearization was also obtained by Jimbo and Sakai. Provided $r = \mu uvp$ holds we can obtain for $x$ the discrete Riccati:

$$x = \frac{\underline{x}(u - pz) + p(u-w)}{\underline{x}z(p^2-1) + p(pw - z)}. \tag{2.15}$$

The equation can be linearized through the Cole–Hopf $x = H/G$ leading to the equation for $G$:

$$\overline{G} + (2\lambda pz - \lambda u - p^2 w)G + \lambda p(z - pu)(pz - w)\underline{G} = 0. \tag{2.16}$$

Jimbo and Sakai, who first obtained this mapping, have identified it as the equation for the $q$-hypergeometric ${}_2\phi_1$.

### 2.4. *The discrete* $\mathrm{P_V}$

From the diagram of the introduction we can see that the asymmetric d-$\mathrm{P_{IV}}$ and the asymmetric $q$-$\mathrm{P_{III}}$ go to the same equation in the coalescence limit. This equation was first identified in [8] where we have shown that it is a discrete form of $\mathrm{P_V}$. Let us first examine the degeneration asymmetric d-$\mathrm{P_{IV}} \to$ d-$\mathrm{P_V}$. We put $X = k + x$, $Y = -k + (y+z)\delta$, $Z = -k + z\delta$, $U = k + r + u\delta$, $V = -k + v\delta$, $W = k - r + w\delta$, $P = p\delta$, $R = r$, $S = -k + s\delta$ and from (2.8) we obtain at the the limit $\delta \to 0$ the equations

$$x\underline{x} = \frac{(y+z-v)(y+z-s)}{(y+p)(y-p)} \tag{2.17a}$$

$$y + \overline{y} = -\frac{\tilde{z}+u}{x/c+1} - \frac{\tilde{z}+w}{xc+1} \tag{2.17b}$$

where the constraint $u + v + w + s = 0$ still holds and we have, moreover, taken $4k^2 - r^2 = 1$, $c = 2k + r$.

The linearization of d-$P_V$ can be obtained from the direct splitting of (2.17), but also from the degeneration of the linearization of asymmetric d-$P_{IV}$. The result is the system

$$\begin{aligned} x &= -\frac{c(y+z-v)}{y+p} \\ \underline{x} &= -\frac{y+z-s}{c(y-p)} \\ y &= -\frac{\tilde{z}+u}{x/c+1} - p \\ \overline{y} &= -\frac{\tilde{z}+w}{xc+1} + p \end{aligned} \tag{2.18}$$

under the linearization constraint

$$u + v + p + \alpha/2 = 0 \tag{2.19}$$

(recall $z = \alpha n + \beta$ and $\tilde{z} = z + \alpha/2$ ). A discrete Riccati is easily obtained from (2.18)

$$x = \frac{\underline{x}c^2(v-p-z) + c(v-s)}{2\underline{x}cp + p + s - z}. \tag{2.20}$$

The linearization of (2.22) can be obtained through a Cole–Hopf $x = H/G$ leading to

$$\overline{G} + ((c^2+1)z + (c^2-1)p - c^2v - s + \alpha)G + c^2(z-v-p)(z-s+p)\underline{G} = 0. \tag{2.21}$$

This equation can be transformed through a gauge tranfsormation $G = \Phi F$, with $\Phi = (v+p-z)\underline{\Phi}$ to

$$(\overline{z} - v - p)\overline{F} - ((c^2+1)z + (c^2-1)p - c^2v - s + \alpha)F + c^2(z-s+p)\underline{F} = 0. \tag{2.22}$$

It can be easily shown that (2.22) is just one of the Gauss relations for contiguous hypergeometric functions. In fact, equation (2.22) is satisfied by $F(1 + (z-v-p)/\alpha, (s-v)/\alpha; 1 + (s-v-2p)/\alpha; 1 - 1/c^2)$ [13].

The relation of the special function solutions of d-$P_V$ to the hypergeometric equation is not at all astonishing. Indeed in [14] we have shown that (2.17) can be obtained from the Schlesinger transformations of $P_{VI}$. This means that the dependent variable of the discrete equation coincides (under the proper choice) with that of the continuous equation. Thus it makes sense to find that the special solutions of the discrete $P_V$ obey the contiguity relations of the function that appears in the special solutions of $P_{VI}$, namely the hypergeometric.

As we explained above, d-$P_V$ in the form of equation (2.17), can be obtained as a degeneration of asymmetric $q$-$P_{III}$. This is in fact how it was first obtained. We shall not go into the details. It is straightforward to check that the linearization of the d-$P_V$ equations obtained from that of the $q$-$P_{III}$ through the coalescence procedure gives the same result as the one obtained above.

The discrete $P_V$ has two possible degenerations to $P_{IV}$ and to asymmetric $P_{II}$. Let us start with the first degeneration.

### 2.5. *The discrete* $\mathrm{P_{IV}}$

This degeneration was first obtained in [8]. Starting from d-$\mathrm{P_V}$, equation (2.17), we introduce the coalescence: $X = x/\delta$, $Y = y$, $U = u/\delta^2$, $V = -u/\delta^2$, $W = w$, $S = s$, $P = p$, $C = -\delta$. At the limit $\delta \to 0$ we obtain

$$x\underline{x} = u\frac{(y+z-s)}{(y+p)(y-p)} \tag{2.23a}$$

$$y+\overline{y} = \frac{u}{x} + \frac{\tilde{z}+w}{x-1} \tag{2.23b}$$

which was shown in [8] to go over to $\mathrm{P_{IV}}$ at the continuous limit. From the linearization equations for d-$\mathrm{P_V}$ we obtain simply

$$x = \frac{u}{y+p} \tag{2.24a}$$

$$\underline{x} = \frac{y+z-s}{y-p} \tag{2.24b}$$

$$y+p = \frac{u}{x} \tag{2.24c}$$

$$\overline{y}-p = \frac{\tilde{z}+w}{x-1} \tag{2.24d}$$

and the linearizability condition $p = s + w - \alpha/2$. The discrete Riccati equation for $x$ now becomes

$$x = \frac{u(\underline{x}-1)}{2p\underline{x}+z-s-p}. \tag{2.25}$$

The linearization of this equation through a Cole–Hopf transformation $x = H/G$ results to the linear equation

$$\overline{G} - (z+\alpha+u-s-p)G + u(z+p-s)\underline{G}. \tag{2.26}$$

This equation is (up to a trivial gauge) the recurrence relation (with respect to the second parameter) for the Kummer confluent hypergeometric $U$ function [13] which is quite reasonable since d-$\mathrm{P_{IV}}$ is related to the continuous $\mathrm{P_V}$ equation [15].

### 2.6. *The asymmetric* d-$\mathrm{P_{II}}$ *(discrete* $\mathrm{P_{II}}$*) equation*

The other degeneration of d-$\mathrm{P_V}$ is towards the asymmetric d-$\mathrm{P_{II}}$, which was shown in [10] to be a discrete form of the $\mathrm{P_{III}}$ equation. Starting from d-$\mathrm{P_V}$, equation (2.17), we put $X = 1+\delta x$, $Y = y$, $Z = \delta z$, $U = -\delta w = -W$, $V = 1+\delta v = -S$, $C = -1-\delta$, $P = 1$, and we obtain at $\delta \to 0$

$$x+\underline{x} = \frac{z+s}{y-1} + \frac{z-s}{y+1} = \frac{2zy+2s}{y^2-1} \tag{2.27a}$$

$$y+\overline{y} = \frac{\tilde{z}-w}{x-1} + \frac{\tilde{z}+w}{x+1} = \frac{2\tilde{z}x-2w}{x^2-1}. \tag{2.27b}$$

The linearization can be obtained from the one of d-$\mathrm{P_V}$ or by direct splitting of (2.29) to

$$x-1 = \frac{z+s}{y-1} \tag{2.28a}$$

$$\underline{x}+1 = \frac{z-s}{y+1} \tag{2.28b}$$

$$y-1 = \frac{\tilde{z}-w}{x-1} \tag{2.28c}$$

$$\overline{y} + 1 = \frac{\tilde{z} + w}{x + 1}. \tag{2.28d}$$

The linearizability/compatibility condition reads $s + w = \alpha/2$ and leads to the discrete Riccati

$$x = \frac{(2 - z - s)\underline{x} + 2 - 2z}{2\underline{x} + 2 - z + s}. \tag{2.29}$$

The linearization is again obtained through $x = H/G$ and results in

$$\overline{G} + (2z + \alpha - 4)G + (z^2 - s^2)\underline{G} = 0. \tag{2.30}$$

This equation is, just like (2.26), a recurrence relation of the Kummer $U$ function [13] (with respect to its first parameter, and up to a simple gauge transformation). As a matter of fact the asymmetric d-$\mathrm{P_{II}}$ (discrete $\mathrm{P_{III}}$) equation can also be obtained [15] from the Schlesinger transformations of the continuous $\mathrm{P_V}$ and is intimately related to the discrete $\mathrm{P_{IV}}$, equation (2.23). The two equations share the same 'Grand Scheme' [16].

### 2.7. *The asymmetric* d-$\mathrm{P_I}$ *equation*

This equation was studied in great detail in [17] where we have shown its relation to the continuous $\mathrm{P_{IV}}$. This asymmetric equation is just another form of d-$\mathrm{P_{II}}$. In the coalescence cascade we presented in the introduction it can be obtained as a degeneration of both d-$\mathrm{P_{IV}}$ and asymmetric d-$\mathrm{P_{II}}$. Let us show how the first limit can be obtained. We start from (2.23) and put: $X = 1 + \delta x/2$, $Y = 1 + \delta y$, $S = 1 + \delta^2 s/2$, $W = \delta^2 w/2$, $P = 1$, $Z = \delta^2 z/2$. At the limit $\delta \to 0$ we find

$$x + \underline{x} = -y + u + \frac{z - s}{y} \tag{2.31a}$$

$$y + \overline{y} = -x + u + \frac{\tilde{z} + w}{x}. \tag{2.31b}$$

The linearization splitting is

$$x = -y + u \tag{2.32a}$$

$$\underline{x} = \frac{z - s}{y} \tag{2.32b}$$

$$y = -x + u \tag{2.32c}$$

$$\overline{y} = \frac{\tilde{z} + w}{x} \tag{2.32d}$$

and the condition reads $s + w = \alpha/2$. Using (2.32) we can obtain a discrete Riccati for $x$:

$$x = u + \frac{s - z}{\underline{x}} \tag{2.33}$$

which linearizes, through $x = H/G$, to

$$\overline{G} - uG + (z - s)\underline{G} = 0 \tag{2.34}$$

i.e. a discrete analogue of the Airy equation which is nothing but a recurrence relation of the parabolic cylinder equation, a fact that is expected, given the relation of asymmetric d-$\mathrm{P_I}$ to $\mathrm{P_{IV}}$.

The asymmetric d-$\mathrm{P_I}$ equation can be also obtained from the asymmetric d-$\mathrm{P_{II}}$ through a coalescence limit. This procedure is essentially the same as the one introduced in [3, 8] for the degeneration of the *symmetric* d-$\mathrm{P_{II}}$ to d-$\mathrm{P_I}$. The linear equation resulting from this coalescence is, of course, the same discrete Airy as in equation (2.34).

## 3. Conclusion

In this paper we have presented the solutions of a class of asymmetric discrete Painlevé equations for special values of parameters for which the equations can be linearized. We have concentrated on the fundamental solutions of this type, namely the ones which can be linearized through a simple Cole–Hopf transformation.

The solutions of the discrete Painlevé equations can then be expressed in terms of the discrete analogue of the special functions i.e. functions belonging to the hypergeometric family in its various disguises.

Solutions of the discrete Painlevé equations can be constructed in terms of special functions for other values of the parameters. The condition for their existence is simply related to the one obtained in the linearizable case. These higher solutions can be expressed in terms of Casorati determinants. The question, as far as most equations presented here are concerned, is open. Only for asymmetric $q$-$\mathrm{P_{III}}$ [18] and asymmetric d-$\mathrm{P_I}$ [17] do we possess the Casorati solutions, involving $q$-hypergeometric and discrete Airy functions, respectively.

Constructing higher solutions is straightforward once the Schlesinger transformations for the d-P at hand are known. Starting from the elementary linearizable solution one can construct the higher ones step by step by iterating the Schlesinger transformations. On the other hand, the advantage of the Casorati solutions lies in the fact that they give a global expression which does not require successive iterations.

Another type of solution that we have not considered at all are the rational ones. It is indeed easy to show that for special values of the parameters (in general different from the ones that lead to linearization) the discrete Painlevé equations do possess rational solutions. It is quite straightforward to obtain the elementary ones and then the higher ones through the application of the Schlesinger transformation, but to organize them in terms of Casorati determinants is a much more difficult task than the equivalent one for special functions. Very few results exist to date in this direction.

In this paper we have concentrated on a particular family of asymmetric discrete Painlevé equations, the ones related to the 'standard' [3] forms and their degenerations through coalescence. It is clear that the asymmetric forms examined here are not the only possible ones. As a recent work of ours has shown [7], many more asymmetric d-P do exist and their classification is far from complete. Once these asymmetric forms are established, the study of their special solutions can be understood without difficulty using the method we developed here.

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